

# A FAMILY OF MARTINGALES GENERATED BY A PROCESS WITH INDEPENDENT INCREMENTS

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ABSTRACT. An explicit procedure to construct a family of martingales generated by a process with independent increments is presented. The main tools are the polynomials that give the relationship between the moments and cumulants, and a set of martingales related to the jumps of the process called Teugels martingales.

## 1. INTRODUCTION

In this work we present an explicit procedure to generate a family of martingales from a process  $X = \{X_t, t \geq 0\}$  with independent increments and continuous in probability. We extend our results exposed in [8], where we dealt with Lévy processes (independent and stationary increments); in that case, the martingales obtained were of the form  $M_t = P(X_t, t)$ , where  $P(x, t)$  is a polynomial in  $x$  and  $t$ , and then they are *time-space* harmonic polynomials relative to  $X$ . Here, the martingales constructed are polynomials on  $X_t$  but, in general, not in  $t$ . Part of the paper is devoted to define the Teugels martingales of a process with independent increments; such martingales, introduced by Nualart and Schoutens [5] for Lévy processes, are a building block of the stochastic calculus with that type of processes.

## 2. INDEPENDENT INCREMENT PROCESSES AND THEIR TEUGELS MARTINGALES

Let  $X = \{X_t, t \geq 0\}$  be a process with independent increments,  $X_0 = 0$ , continuous in probability and cadlag; such processes are also called additive processes, and we will indistinctly use both names. Moreover, assume that  $X_t$  is centered, and has moment of all orders. It is well known that the law of  $X_t$  is infinitely divisible for all  $t \geq 0$ . Let  $\sigma_t^2$  the variance of the gaussian part of  $X_t$ , and  $\nu_t$  its Lévy measure; for all these notions we refer to Sato [6] or Skorohod [7].

Denote by  $\tilde{\nu}$  the (unique) measure on  $\mathcal{B}((0, \infty) \times \mathbb{R}_0)$  defined by

$$\tilde{\nu}((0, t] \times B) = \nu_t(B), \quad B \in \mathcal{B}(\mathbb{R}_0),$$

where  $\mathbb{R}_0 = \mathbb{R} - \{0\}$ . By an standard approximation argument, we have that for a measurable function  $f : \mathbb{R}_0 \rightarrow \mathbb{R}$ , and for every  $t > 0$ ,

$$\iint_{(0, t] \times \mathbb{R}_0} |f(x)| \tilde{\nu}(ds, dx) < \infty \quad \Longleftrightarrow \quad \int_{\mathbb{R}_0} |f(x)| \nu_t(dx) < \infty,$$

and in this case,

$$\iint_{(0, t] \times \mathbb{R}_0} f(x) \tilde{\nu}(ds, dx) = \int_{\mathbb{R}_0} f(x) \nu_t(dx).$$

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Note that since for every  $t \geq 0$ ,  $\nu_t$  is a Lévy measure, then  $\tilde{\nu}$  is  $\sigma$ -finite. To prove this, observe that  $\nu_t(\{|x| > 1\}) < \infty$ , and  $\nu_t((1/(n+1), 1/n]) < \infty$  and  $\nu_t([-1/n, -1/(n+1))) < \infty$ ,  $n \geq 1$ . So there is a numerable partition of  $\mathbb{R}_0$  with sets of finite  $\nu_t$  measure,  $\forall t > 0$ . Then, we can construct a numerable partition of  $(0, \infty) \times \mathbb{R}_0$ , each set with finite  $\tilde{\nu}$ -measure.

Write

$$N(C) = \#\{t : (t, \Delta X_t) \in C\}, \quad C \in \mathcal{B}((0, \infty) \times \mathbb{R}_0),$$

the jump measure of the process, where  $\Delta X_t = X_t - X_{t-}$ . It is a Poisson random measure on  $(0, \infty) \times \mathbb{R}_0$  with intensity measure  $\tilde{\nu}$  (Sato [6, Theorem 19.2]). Define the compensated jump measure

$$d\tilde{N}(t, x) = dN(t, x) - d\tilde{\nu}(t, x).$$

The process admits the Lévy-Itô representation

$$X_t = G_t + \iint_{(0, t] \times \mathbb{R}_0} x d\tilde{N}(t, x), \quad (2)$$

where  $\{G_t, t \geq 0\}$  is a centered continuous Gaussian process with independent increments and variance  $\mathbb{E}[G_t^2] = \sigma_t^2$ .

It is also well known the relationship between the moments of an infinitely divisible law and the *moments* of its Lévy measure (see Sato [6, Theorem 25.4]). In our case, as the process has moments of all orders, for all  $t \geq 0$ ,

$$\int_{\{|x| > 1\}} |x| \nu_t(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}_0} |x|^n \nu_t(dx) < \infty, \quad \forall n \geq 2.$$

Write

$$F_2(t) = \sigma_t^2 + \int_{\mathbb{R}_0} x^2 \nu_t(dx) \quad \text{and} \quad F_n(t) = \int_{\mathbb{R}_0} x^n \nu_t(dx), \quad n \geq 3. \quad (1)$$

Since  $\int_{\{|x| > 1\}} |x| \nu_t(dx) < \infty$  and  $\mathbb{E}[X_t] = 0$ , the characteristic function of  $X_t$  can be written as

$$\phi_t(u) = \exp \left\{ -\frac{1}{2} \sigma_t^2 u^2 + \int_{\mathbb{R}_0} (e^{iux} - 1 - iux) \nu_t(dx) \right\}.$$

It is deduced that for  $n \geq 2$ ,  $F_n(t)$  is the cumulant of order  $n$  of  $X_t$  (for  $n = 2$ ,  $\mathbb{E}[X_t^2] = F_2(t)$ ). Also  $\sigma_t^2$  is continuous and increasing (Sato [6, Theorem 9.8]).

**Proposition 1.** *The functions  $F_n(t)$ ,  $n \geq 2$ , are continuous and have finite variation on finite intervals, and for  $n$  even, they are increasing.*

*Proof.*

Consider  $0 < u < t < v$ , and write  $U = [u, v]$ . From the continuity in probability of  $X$ ,

$$\lim_{s \rightarrow t, s \in U} X_s^n = X_t^n, \quad \text{in probability.}$$

Moreover,  $\forall s \in U$ ,  $|X_s| \leq \sup_{r \in U} |X_r|$ , and since  $X$  is a martingale, by Doob's inequality,

$$\mathbb{E} \left[ \sup_{r \in U} |X_r|^n \right] \leq C \sup_{r \in U} \mathbb{E}[|X_r|^n] \leq C \mathbb{E}[|X_v|^n] < \infty.$$

So by dominated convergence it follows that the function  $t \mapsto \mathbb{E}[X_t^n]$  is continuous. Since the cumulants are polynomials of the moments, it is deduced the continuity of all functions  $F_n(t)$ .

To prove that  $F_n(t)$  has finite variation on finite intervals, consider a partition of  $[0, t]$ :  $0 < t_0 < \dots < t_k = t$ . Then

$$\begin{aligned} \sum_{j=1}^k |F_n(t_j) - F_n(t_{j-1})| &= \sum_{j=1}^k \left| \iint_{(t_{j-1}, t_j] \times \mathbb{R}_0} x^n \tilde{\nu}(ds, dx) \right| \\ &\leq \sum_{j=1}^k \iint_{(t_{j-1}, t_j] \times \mathbb{R}_0} |x|^n \tilde{\nu}(ds, dx) = \iint_{(0, t] \times \mathbb{R}_0} |x|^n \tilde{\nu}(ds, dx) < \infty. \quad \blacksquare \end{aligned}$$

Consider the *variations* of the process  $X$  (see Meyer [4]):

$$\begin{aligned} X_t^{(1)} &= X_t, \\ X_t^{(2)} &= [X, X]_t = \sigma_t^2 + \sum_{0 < s \leq t} (\Delta X_s)^2 \\ X_t^{(n)} &= \sum_{0 < s \leq t} (\Delta X_s)^n, \quad n \geq 3. \end{aligned}$$

By Kyprianou [2, Theorem 2.7], for  $n \geq 3$  (the case  $n = 2$  is similar), the characteristic function of  $X^{(n)}$  is

$$\exp \left\{ \iint_{(0, t] \times \mathbb{R}_0} (e^{iux^n} - 1) \tilde{\nu}(ds, dx) \right\} = \exp \left\{ \int_{\mathbb{R}_0} (e^{iux} - 1) \nu_t^{(n)}(dx) \right\},$$

where  $\nu_t^{(n)}$  is the measure image of  $\nu_t$  by the function  $x \mapsto x^n$ , which is a Lévy measure. So  $X^{(n)}$  has independent increments. Also by Kyprianou [2, Theorem 2.7], for  $n \geq 2$ ,

$$\mathbb{E}[X_t^{(n)}] = F_n(t) \quad \text{and} \quad \mathbb{E}[(X_t^{(n)})^2] = F_{2n}(t) + (F_n(t))^2.$$

Therefore, combining the independence of the increments and the continuity of  $F_n(t)$ , it is deduced that  $X^{(n)}$  is continuous in probability.

By Proposition 1,  $F_n(t)$  has finite variation on finite intervals. Hence, the process

$$X_t^{(n)} = F_n(t) + (X_t^{(n)} - F_n(t))$$

is a semimartingale.

The Teugels martingales introduced by Nualart and Schoutens [5] for Lévy processes can be extended to additive processes. In the same way as in [5], these martingales are obtained centering the processes  $X^{(n)}$ :

$$\begin{aligned} Y_t^{(1)} &= X_t, \\ Y_t^{(n)} &= X_t^{(n)} - F_n(t), \quad n \geq 2, \end{aligned}$$

They are square integrable martingales with optional quadratic covariation

$$[Y^{(n)}, Y^{(m)}]_t = X^{(n+m)}_t,$$

and, since  $F_{2n}(t)$  is increasing, the predictable quadratic variation of  $Y^{(n)}$  is

$$\langle Y^{(n)} \rangle_t = F_{2n}(t).$$

## 3. THE POLYNOMIALS OF CUMULANTS

The formal expression

$$\exp \left\{ \sum_{n=1}^{\infty} \kappa_n \frac{u^n}{n!} \right\} = \sum_{n=0}^{\infty} \mu_n \frac{u^n}{n!}. \quad (3)$$

relates the sequences of numbers  $\{\kappa_n, n \geq 1\}$  and  $\{\mu_n, n \geq 0\}$ . When we consider a random variable  $Z$  with moment generating function in some open interval containing 0, then both series converge in a neighborhood of 0, and (3) is the relationship between the moment generating function,  $\psi(u) = \mathbb{E}[e^{uZ}]$ , and the cumulant generating function,  $\log \psi(u)$ . Moreover,  $\mu_n$  (respectively,  $\kappa_n$ ) is the moment (respectively, the cumulant) of order  $n$  of  $Z$ , and the well known relationship between moments and cumulants can be deduced from (3). The first three are

$$\begin{aligned} \mu_1 &= \kappa_1, \\ \mu_2 &= \kappa_1^2 + \kappa_2, \\ \mu_3 &= \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3, \dots \end{aligned}$$

If the random variable  $Z$  has only finite moments up to order  $n$ , the corresponding relationship is true up to this order.

There is a general explicit expression of the moments in terms of cumulants in Kendall and Stuart [1], or formulas involving the partitions of a set, see McCullagh [3]. In general,  $\mu_n$  is a polynomial of  $\kappa_1, \dots, \kappa_n$ , called Kendall polynomial. Denote by  $\Gamma_n(x_1, \dots, x_n)$ ,  $n \geq 1$ , this polynomial, that is, we have

$$\mu_n = \Gamma_n(\kappa_1, \dots, \kappa_n).$$

Also write  $\Gamma_0 = 1$ . These polynomials enjoy very interesting properties, as the recurrence formula that follows from Stanley [9, Proposition 5.1.7]:

$$\Gamma_{n+1}(x_1, \dots, x_{n+1}) = \sum_{j=0}^n \binom{n}{j} \Gamma_j(x_1, \dots, x_j) x_{n+1-j}. \quad (4)$$

We also have

$$\frac{\partial \Gamma_n(x_1, \dots, x_n)}{\partial x_j} = \binom{n}{j} \Gamma_{n-j}(x_1, \dots, x_{n-j}), \quad j = 1, \dots, n. \quad (5)$$

Computing the Taylor expansion of  $\Gamma_n(x_1 + y, x_2, \dots, x_n)$  at  $y = 0$ , we get the following expression that we will need later:

$$\Gamma_n(x_1 + y, x_2, \dots, x_n) = \sum_{j=0}^n \binom{n}{j} \Gamma_{n-j}(x_1, \dots, x_{n-j}) y^j. \quad (6)$$

Interchanging the role of  $x_1$  and  $y$ , and evaluating the function at 0 we obtain

$$\Gamma_n(x_1, x_2, \dots, x_n) = \sum_{j=0}^n \binom{n}{j} \Gamma_{n-j}(0, x_2, \dots, x_{n-j}) x_1^j. \quad (7)$$

## 4. A FAMILY OF MARTINGALES RELATIVE TO THE ADDITIVE PROCESS

The main result of the paper is the following Theorem:

**Theorem 1.** *Let  $X$  be a centered additive process with finite moments of all orders. Then the process*

$$M_t^{(n)} = \Gamma_n(X_t, -F_2(t), \dots, -F_n(t))$$

*is a martingale.*

*Proof.*

Let  $n \geq 2$ . We apply the multidimensional Itô formula to the semimartingales  $X_t, F_2(t), \dots, F_n(t)$ . By Proposition 1, the functions  $F_2(t), \dots, F_n(t)$  and  $\sigma_t^2$  are continuous and of finite variation. From (5) and the fact that  $[X, X]_t^c = \sigma_t^2$  and  $[F_j, F_j]_t^c = 0$ , we have

$$\begin{aligned} M_t^{(n)} &= n \int_0^t M_{s-}^{(n-1)} dX_s - \sum_{j=2}^n \binom{n}{j} \int_0^t M_s^{(n-j)} dF_j(s) + \frac{1}{2} n(n-1) \int_0^t M_s^{(n-2)} d(\sigma_s^2) \\ &\quad + \sum_{0 < s \leq t} \left( \Gamma_n(X_{s-} + \Delta X_s, -F_2(s), \dots, -F_n(s)) - \Gamma_n(X_{s-}, -F_2(s), \dots, -F_n(s)) \right) \\ &\quad - n \Delta X_s \Gamma_{n-1}(X_{s-}, -F_2(s), \dots, -F_n(s)) \end{aligned}$$

Applying (6),

$$\Gamma_n(X_{s-} + \Delta X_s, -F_2(s), \dots, -F_n(s)) = \sum_{j=0}^n \binom{n}{j} M_{s-}^{(n-j)} (\Delta X_s)^j.$$

Then, the jumps part given in the expression of  $M_t^{(n)}$  is

$$\begin{aligned} \sum_{0 < s \leq t} \sum_{j=2}^n \binom{n}{j} M_{s-}^{(n-j)} (\Delta X_s)^j &= \sum_{j=2}^n \binom{n}{j} \int_0^t M_{s-}^{(n-j)} dX_s^{(j)} - \binom{n}{2} \int_0^t M_s^{(n-2)} d(\sigma_s^2) \\ &= \sum_{j=2}^n \binom{n}{j} \int_0^t M_{s-}^{(n-j)} d(Y_s^{(j)} + F_j(s)) - \binom{n}{2} \int_0^t M_s^{(n-2)} d(\sigma_s^2). \end{aligned}$$

Therefore,

$$M_t^{(n)} = \sum_{j=1}^n \binom{n}{j} \int_0^t M_{s-}^{(n-j)} dY_s^{(j)}. \quad (8)$$

Moreover,  $(M_t^{(k)})^2$  is a polynomial in  $X_t, F_2(t), \dots, F_k(t)$ . Taking expectations and using the relationship between moments and cumulants, and that the cumulants of  $X_t$  are  $F_n(t)$ ,  $n \geq 2$ , we obtain that

$$E[(M_t^{(k)})^2] = P(F_2(t), \dots, F_{2k}(t)),$$

for a suitable polynomial  $P$ . Then, for every  $t \geq 0$ , we have

$$\mathbb{E} \left[ \int_0^t (M_{s-}^{(k)})^2 d\langle Y^{(j)} \rangle_s \right] = \int_0^t \mathbb{E}[(M_{s-}^{(k)})^2] dF_{2j}(s) = \int_0^t P(F_2(s), \dots, F_{2k}(s)) dF_{2j}(s) < \infty,$$

So all the stochastic integrals on the right hand side of (8) are martingales. ■

**Remark 1.** *It is worth to note that it follows from the preceding Theorem that the function*

$$g_n(x, t) = \Gamma_n(x, -F_2(t), \dots, -F_n(t))$$

*is a time-space harmonic function with respect to  $X_t$ . By (7)*

$$g_n(x, t) = \sum_{j=0}^n \Gamma_{n-j}(0, -F_2(t), \dots, F_n(t)) x^j.$$

*In general,  $g_n(x, t)$  is a polynomial in  $x$ . If  $F_n(t)$ ,  $n \geq 2$ , are polynomials in  $t$ , then  $g_n(x, t)$  is a time-space harmonic polynomial; this happens for all Lévy processes with moments of all orders, and for some additive process; see the example below.*

**Example.** Let  $\Lambda(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous increasing function, and  $J$  be a Poisson random measure on  $\mathbb{R}_+$  with intensity measure  $\mu(A) = \int_A \Lambda(dt)$ ,  $A \in \mathcal{B}(\mathbb{R}_+)$ . Then the process  $X = \{X_t, t \geq 0\}$  defined pathwise,

$$X_t(\omega) = \int_0^t J(ds, \omega) - \Lambda(t)$$

is an additive process; it is a Cox process with deterministic hazard function  $\Lambda(t)$ . From the characteristic function of  $X_t$  we deduce that the Lévy measure is

$$\nu_t(dx) = \Lambda(t) \delta_1(dx),$$

where  $\delta_1$  is a Dirac delta measure concentrated in the point 1. Hence,

$$F_n(t) = \Lambda(t), \quad n \geq 2.$$

Note that the conditions that we have assumed on  $\Lambda$  are necessary to obtain an additive process, but it is not necessary (though not very restrictive) to assume that  $\Lambda$  is absolutely continuous with respect to the Lebesgue measure.

The function defined in Remark 1 is

$$g_n(x, t) = \Gamma_n(x, -\Lambda(t), \dots, -\Lambda(t)).$$

Hence, when  $\Lambda(t)$  is a polynomial,  $g_n(x, t)$  is a time-space harmonic polynomial.

Denote by  $\overline{C}_n(x, t)$  the Charlier polynomial with leading coefficient equal to 1, then (see [8])

$$g_n(x, t) = \sum_{j=1}^n \lambda_j^{(n)} \overline{C}_j(x, \Lambda(t)),$$

where  $\lambda_1^{(n)} = 1$  and

$$\lambda_{k+1}^{(n)} = \sum_{j=k}^{n-1} \binom{n}{j} \lambda_k^{(j)}, \quad k = 1, \dots, n-1.$$

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